APPLICATION OF LAPLACE TRANSFORM IN SCIENCE AND ENGINEERING FIELDS

ABSTRACT

The Laplace Transform is a widely used integral transform in mathematics with many applications in Science and Engineering. The aim of this project is to study about the applications of Laplace transform in various fields. In the first part of this project, the basic definitions and properties of Laplace transform are presented. In the later part applications of Laplace transform in the areas of Science and Engineering fields such as solving population growth and decay problems, mechanics, as well as Control, electrical and Mechanical engineering are discussed.

CHAPTER 1

INTRODUCTION

Laplace Transform is used to simplify calculations in system modeling, where large numbers of differential equations are used. Laplace Transform is widely used by engineers to solve quickly differential equations occurring in the analysis of electronic circuits.

A Laplace transform is an extremely diverse function that can transform a real function of time t to one in the complex plane s, referred to as the frequency domain. The Laplace transform is a complex transform of a complex variable. The method of solving linear differential equations was originally published by Laplace nearly one hundred and fifty years ago (1812), and has been used by mathematicians alongside the other methods as a regular training in higher mathematics for a good many years.

The primary use of this transform is to change an ordinary differential equation in a real domain into an algebraic equation in the complex domain, making the equation much easier to solve. The subsequent solution that is found by solving the algebraic equation is then taken and inverted by use of the inverse Laplace transform, acquiring a solution for the original differential equation, or ODE.

The concepts of Laplace transform are applied in area of science and technology. In solving problems related to their fields, one usually encounters problems on time invariant, differential equations, time frequency domain for non periodic wave forms.

In this project, various applications are used in solving various problems in real life using “Laplace Transform” and their examples are also being discussed.

In Chapter II, definition of Laplace Transform and their properties are discussed.

In Chapter III, Laplace Transform used to solve population growth, decay problems. And application of Laplace Transform in Mechanics is also discussed.

In Chapter IV, Application of Laplace transform in engineering field such as Automatic control, electrical engineering and Mechanical engineering are explained.

CHAPTER 2

PRELIMINARIES

Definition: Laplace Transform

Laplace transform is the integral transform of the given derivative function with real variable 't' to convert into complex function with variable 's'. Let f (t) be a function of t (t > 0), then the integral $\int_0^\infty e^{-st}$f (t) dt is called Laplace transform of f (t).

We denote it as L [f (t)] or F(s).

i.e) L [f (t)] = $\int_0^\infty e^{-st}$ f (t) dt = F(s)

SOME FORMULAE OF LAPLACE TRANSFORM

1. L [δ(t)] = 1
2. L [u(t)] = $\frac{1}{s}$
3. L [t] = $\frac{1}{s^2}$
4. L [t^2] = $\frac{2}{s^3}$
5. L [t^3] = $\frac{6}{s^4}$
6. L [t^n] = $\frac{n!}{s^{n+1}}$
7. L [$e^{-\alpha t}$] = $\frac{1}{s+\alpha}$
8. L [$e^{\alpha t}$] = $\frac{1}{s-\alpha}$
9. L [t$e^{-\alpha t}$] = $\frac{1}{(s+\alpha)^2}$
10. L [t$e^{\alpha t}$] = $\frac{1}{(s-\alpha)^2}$
11. L [$t^n e^{-\alpha t}$] = $\frac{n!}{(s+\alpha)^{n+1}}$
12. L [cosωt] = $\frac{s}{s^n+\omega^2}$
13. L [sinωt] = $\frac{\omega}{s^2+\omega^2}$
14. L [$e^{-\alpha t}$sinωt]=$\frac{\omega}{(s+\alpha)^2+\omega^2}$
15. L [$e^{\alpha t}$cosωt] = $\frac{s+\alpha}{(s+\alpha)^2+\omega^2}$
16. L [sinh αt] = $\frac{\alpha}{s^2-\alpha^2}$
17. L [cosh αt] = $\frac{s}{s^2-\alpha^2}$

PROPERTIES OF LAPLACE TRANSFORM

Linearity Property: If f (t) and g(t) are any two functions of **t** and **α,β** are any two constant then,

$$L[\alpha f(t) + \beta g(t)] = \alpha L[f(t)] + \beta L[g(t)]$$

Shifting Property:

If L [f (t)] = F(s), then L [e^{at}f (t)] = F(s-a).

Multiplication by t^n Property:

L [f (t)]= F (s), then

$$L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}[F(s)]$$

Laplace Transform of Derivative:

If L [f (t)] = F (s), then

For nth order derivative,

$$L[f^n(t)] = s^n F(s) - s^{n-1} f'(0) - s^{n-3} f''(0) - \cdots - f^{n-1}(0).$$

Laplace Transform of Bessel's function:

$$L[J_0(t)] = \frac{1}{\sqrt{s^2+1}}\text{, where}$$

$$J_0(t) = \sum_{k=0}^{\infty}(-1)^k \frac{\left(\frac{1}{4}t^2\right)^K}{(K!)}$$ is called Bessel's function.

CHAPTER 3

APPLICATION OF LAPLACE TRANSFORMATION IN THE FIELD OF SCIENCE

3.1 POPULATION GROWTH PROBLEMS

The population growth (growth of a plant, or a cell, or an organ, or a species) is governed by the first order linear ordinary differential equation.

$$\frac{dN}{dt} \alpha\ N$$

$$\frac{dN}{dt} = KN \qquad (3.1.1)$$

With initial condition as

$$N\ (t_0) = N_0 \qquad (3.1.2)$$

Where K is a positive real number, N is the amount of population at time t and N_0 is the initial population at time t_0.

Equation (3.1.1) is known as the Malthusian law of population growth.

Applying the Laplace transform on both sides (3.1.1), we have

$$L\left\{\frac{dN}{dt}\right\} = KL\{N(t)\} \qquad (3.1.3)$$

Now applying the property, Laplace transform of derivative of function, on (3.1.3), we have

$$sL\{N(t)\} – N(0) = KL\{N(t)\} \qquad (3.1.4)$$

Using (3.1.2) in (3.1.3) and on simplifications, we have

$$(s – K)\ L\ \{N(t)\} = N_0$$

$$L\ \{N(t)\} = N_0 / (s - K) \qquad (3.1.5)$$

Operating inverse Laplace transform on both sides of (3.1.5), we have

$$N\ (t) = L^{-1}\ \{ N_0 / (s – K)\}$$

$$N(t) = N_0L^{-1}\{(\ s - K)\}$$

$N(t) = N_0 e^{Kt}$ (3.1.6)

which is the required amount of the population at time .

Example 1:

The population of a city grows at a rate proportional to the number of people presently living in the city. If after two years, the population has doubled , and after three years the population is 20,000 , estimate the number of people initially living in the city.

Given, t =2, N=$2N_0$, so using this in (3.1.6), we have

$$2N_0 = N_0 e^{2k}$$

$$e^{2k} = 2$$

Taking logarithm on both sides, we get

$$K = \frac{1}{2} log_e 2 = 0.347 \quad (3.1.7)$$

Now using the condition a t=3, N=20,000,in(3.1.6), we have

$$20{,}000 = N_0 e^{3k} \quad (3.1.8)$$

Putting the value of K from (3.1.7) in (3.1.8),we have

$$20{,}000 = N_0 e^{3\times 0.347}$$

$$20{,}000 = 2.832 N_0$$

$$N_0 \simeq 7062 \quad (3.1.9)$$

Which are the required number of people initially living in the city.

3.2 LAPLACE TRANSFORM IN NUCLEAR PHYSICS (Decay problem):

The following example is based on concepts from nuclear physics. consider the following first order linear differential equation

$$\frac{dN}{dt} \alpha\ N$$

$$\frac{dN}{dt} = -KN \quad (3.2.1)$$

This equation is the fundamental relationship describing radioactivity decay, where N = N(t) represents the number of undecayed atoms remaining in a sample of a radioactivity isotope at time **t** and **K** is the decay constant.

In equation (3.2.1), the negative sign in the R.H.S. is taken because the mass of the substance is decreasing with time and so the derivative $\frac{dN}{dt}$ must be negative.

We can use the Laplace transform to solve the equation.

Rearranging the above equation (3.2.1), we get

$$\frac{dN}{dt} + KN = 0$$

Taking Laplace transform on both sides,

$$L[\frac{dN}{dt}] + KL[N] = 0$$

$$s\,L[N] - N(0) + KL[N] = 0$$

(here $N(0) = N_o$ = undecay atoms at initial point

$$L[N] = \frac{N_o}{s+K}$$

Now, taking inverse Laplace transform on both sides, we get

$$N(t) = L^{-1}[\frac{N_o}{s+K}]$$

$$N(t) = N_o e^{-Kt} \qquad (3.2.2)$$

Which is the number of undecayed atoms at time ‘t’.

Example 1:

A radioactive substance is known to decay at a rate proportional to the amount present. If initially there is 100 milligram of the radioactive substance present and after two hours it is observed that the radioactive substance has lost 10 percent of its original mass, find the half life of the radioactive substance.

Given, at t=0, N=N_0 =100,

Using (3.2.2)

$$N(t) = 100e^{-kt} \qquad (3.2.3)$$

Now at t = 2, the radioactive substance has lost 10 percent of its original mass 100 mg so N =100 – 10 =90, using this in (3.2.3), we have

$$90=100e^{-2k}$$

$$e^{-2k} = 0.90$$

$$K = -\frac{1}{2} log_e 0.90 = 0.05268 \quad (3.2.4)$$

We required t when N = $\frac{N_0}{2}$ =$\frac{100}{2} = 50$ so from (3.2.3),we have

$$50= 100e^{-kt} \quad (3.2.5)$$

Putting the value of K from (3.2.4) in (3.2.5),we have

$$50 = 100e^{-0.05268t}$$

$$e^{-0.05268t} = 0.50$$

$$t = -\frac{1}{0.05268} 0.50$$

$$t = 13.157 hours \quad (3.2.6)$$

Which is the required half –time of the radioactive substance.

Example 2

A phosphorus substance is known to decay at a rate proportional to the amount present . If initially there is 1000 milligrams of the phosphorus substance present after five hours it is observed that the phosphorus substance has lost 20 percent of its original mass, find the life of the phosphorus substance.

Given, at t =0, N=N_o=1000,

From (3.2.2) , we have

$$\text{N(t)} = 1000e^{-Kt} \quad (3.2.7)$$

Now at t=5, the phosphorus substance has lost 20 percent of its original mass 1000mg so N(5) = 1000 $-200 = 800$, using this in (3.2.7) , we have

$$800 = 1000e^{-5K}$$

$$0.8 = e^{-5K}$$

$$K = -\frac{1}{5} log_e (0.8)$$

$$K = 0.04462871026 \tag{3.2.8}$$

Here we want to find time N = $\frac{N_0}{2}$ =500 so from (3.2.7), we have

$$500 = 1000e^{-(-0.04462871026)t}$$

$$t = \frac{log (0.5)}{-0.04462871026}$$

$$t = 15.5314186$$

15.53 hours required half time of the phosphorus substance.

3.3 APPLICATION IN MECHANICS:

Example 1:

When a raindrop falls, it increases in size, so its mass at time t is a function of t, m(t). The rate of growth of mass is km(t) for some positive constant k . When we apply Newton's law of motion in the rain drop, we get (mv)′=gm, where v is the velocity of the raindrop (directed downward) and g is the acceleration due to gravity. The terminal velocity of the raindrop is $v(t)$. Find an expression for the terminal velocity in terms of g and k.

Assume that the raindrop begins at rest, so that v(0)=0.

$$\frac{dm}{dt} \alpha\ km$$

The rate of growth ,$\frac{dm}{dt} = km$.

From applying Newton's Law, (mv)′ =gm

(mv)′ = mv′+vm′ (From Product Rule)

mv′+vm′= gm $\left(substituing\ \frac{dm}{dt} = m' = km\right)$

mv′+v(km)= gm

The equation becomes v′+kv = g

$$\frac{dv}{dt} + kv = g \qquad \text{with initial condition } v(0) = 0$$

Take the Laplace transform as usual

$$\mathrm{L}\left(\frac{dv}{dt}\right) + L(kv) \quad = L(g)$$

$$sL(v) - v(0) + k\,L(v) = L(g)$$

$$sv(s) - v_o + kv(s) = \frac{g}{s}$$

$$v(s) = \frac{g + sv_o}{s(s+k)}$$

$$v(s) = \frac{g}{k}\left(\frac{1}{s}\right) + \left(v_o - \frac{g}{k}\right)\left(\frac{1}{s+k}\right)$$

Taking the inverse Laplace transform

$$v(t) = \frac{g}{k} + \left(v_o - \frac{g}{k}\right)e^{-kt}$$

at $v(0) = v_o = 0$

$$v(t) = \frac{g}{k}(1 - e^{-kt}) \qquad (k>0)$$

since k >0 as t$\rightarrow \infty, e^{-kt} \rightarrow 0\ and\ thus\ v(t) = \frac{g}{k}$

Example 2:

A ball with mass m is projected vertically upwards from earth's surface with a positive velocity v_o. we assume the forces acting on the ball are the force of gravity and a retarding force of air resistance with direction opposite to the direction of motion and with magnitude p |v(t)|, where p is a positive constant and v(t) is the velocity of the ball at time t. in both the ascent and the decent, the total force acting on the ball is -pv – mg. during ascent, v(t) is negative and the resistance acts upwards. So, by newton's second law, the equation of motion is

$$mv' = -pv - mg$$

To find the velocity of the ball at time 't'

$$mv' = -pv - mg$$

$$mv' + pv + mg = 0$$

divide by ‘m’,

$$v' + \frac{p}{m} v + g = 0$$

Taking Laplace transform on both sides,

$$S\, v(t) - v(0) + \frac{p}{m} v(t) + \frac{g}{s} = 0$$

Here v(t) = v(s) and v(0) = v_0

$$v(s)\left(s + \frac{p}{m}\right) = v_0 - \frac{g}{s}$$

$$v(s) = \frac{v_0}{s + \frac{p}{m}} - \frac{g}{s\left(s + \frac{p}{m}\right)}$$

$$v(s) = \frac{v_0}{s + \frac{p}{m}} - \frac{mg}{p}\left[\frac{1}{s} - \frac{1}{\left(s + \frac{p}{m}\right)}\right]$$

$$v(s) = -\frac{mg}{p}\left(\frac{1}{s}\right) + \frac{v_0}{s + \frac{p}{m}} + \frac{mg}{p\left(s + \frac{p}{m}\right)}$$

$$v(s) = -\frac{mg}{p}\left(\frac{1}{s}\right) + \left(v_0 + \frac{mg}{p}\right)\left(\frac{1}{s + \frac{p}{m}}\right)$$

Taking inverse Laplace transform on both sides,

$$L^{-1}(v') = -\frac{mg}{p}\left(L^{-1}\left(\frac{1}{s}\right) + v_0 + \frac{mg}{p}\, L^{-1}\left(\frac{1}{s + \frac{p}{m}}\right)\right)$$

$$v(t) = \left(v_0 + \frac{mg}{p}\right) e^{-\frac{pt}{m}} - \frac{mg}{p}$$

This equation is the velocity of the ball at time ‘t’.

CHAPTER 4

APPLICATION OF LAPLACE TRANSFORMATION IN THE FIELD OF ENGINEERING

4.1 THEORY OF AUTOMATIC CONTROL:

Suppose that a missile **M** is tracking down enemy aircraft. If at time **t** the enemy turns through some angle **Ө(t),** the M must also turns through this angle, if it is to be catch with and destroy it. If a mass was aboard **M**, he could operate.

Some Steering mechanism to produce required turns, but since the missile is must be accomplish automatically. To do for a man eyes, such as a radar beam which must be taken by **M**, we also need something providing a substitute for a man's hands which will turn a shaft through some angle in order to produce the desire turn.

In this application, let us assume that the desired angle of turn as indicated by the radar is αt. Also let **Ө (t)** denotes the angle if turn of the shaft at time **t**. because the things are happening so fast we must except to have a error between two.

error = ɵ (t) – αt

i.e. the existence of the error must be signaled back to the shaft, so that a compensating efforts or torque be produced. If the error is large the torque needed will be large. So that requires torque is proportional to the error

$$\text{TORQUE} = \sum_{i=1}^{n} m_i r_i^2 \times \frac{d^2\theta}{dt^2}$$

$\text{TORQUE} = I\,\frac{d^2\theta}{dt^2}$ i.e. M.I. multiplied by angular acceleration.

Where, I = moment of inertia ,

$\frac{d^2\theta}{dt^2}$ = angular acceleration

Since torque is directly proportional to error, I $\frac{d^2\theta}{dt^2}$ α [Θ(t) – αt]

I$\frac{d^2\theta}{dt^2}$ = - k [Θ(t) – αt] (4.1.1)

Where k > 0

The minus sign is used because if the error is positive then the torque must be opposite it. While the error is negative the torque must be positive.

Assuming that the initial angle and angular velocity are zero as possible conditions.

i.e.Θ(0) = 0, and Θ´(0) = 0

from equation (4.1.1) `

$$I\frac{d^2\theta}{dt^2} = - k\Theta(t) + k\,\alpha t$$

$$\frac{d^2\theta}{dt^2} = -\frac{k}{l}\theta(t) + \frac{k}{l}\alpha t$$

$$\frac{d^2\theta}{dt^2} + \frac{k}{l}\theta(t) = \frac{k}{l}\alpha t$$

Taking Laplace Transform on both sides,

$$L\left[\frac{d^2\theta}{dt^2}\right] + \frac{k}{l}L[\theta(t)] = \frac{k}{l}L[\alpha t]$$

$$s^2L[\theta] - s\theta(0) - \theta'(0) + \frac{k}{l}L[\theta] = \frac{k}{l}\alpha\frac{1}{s^2}$$

using laplace transform of derivative, L[$f^n(t)] = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0)$

$$\left(s^2 + \frac{k}{l}\right)L[\theta] = \frac{k}{l}\alpha\frac{1}{s^2}$$

using Θ(0) = 0, and Θ´(0) = 0

$$L[\theta] = \frac{k}{l}\left[\frac{\alpha}{s^2\left(s^2+\frac{k}{l}\right)}\right]$$

Taking Inverse Laplace Transform on both sides,

$$\Theta = \frac{k}{l}L^{-1}\left[\frac{\alpha}{s^2\left(s^2+\frac{k}{l}\right)}\right]$$

$$\Theta = \frac{k\alpha}{l}\,[L^{-1}\left(\frac{1}{s^2}\right) * L^{-1}\left(\frac{1}{s^2+\frac{k}{l}}\right)]$$

using convolution theorem,

$$L^{-1}\,[\emptyset_1(s).\emptyset_2(s)] = \int_0^t f_1(u).\,f_2\ (t - u)\,du$$

Here $\quad f_1(u) = L^{-1}\left(\frac{1}{s^2}\right) \quad ; \quad f_2(t-u) = L^{-1}\left(\frac{1}{s^2+\frac{k}{l}}\right)$

$$L^{-1}\left(\frac{1}{s^2}\right) = u \quad , \quad L^{-1}\left(\frac{1}{s^2+\frac{k}{l}}\right) = \frac{1}{\sqrt{\frac{k}{l}}}\ \sin\sqrt{\frac{k}{l}}\,(t - u)$$

Substitute in above equation,

$$\Theta = \frac{k\alpha}{l}\int_0^t u\,\frac{1}{\sqrt{\frac{k}{l}}}\quad \sin\sqrt{\frac{k}{l}}\,(t - u)\,du$$

$$\Theta = \frac{\frac{k\alpha}{l}}{\sqrt{\frac{k}{l}}}\ \int_0^t u\,.\sin\sqrt{\frac{k}{l}}\,(t - u)\,du$$

$$\Theta = \alpha\,\sqrt{\frac{k}{l}}\ \int_0^t u\,.\ \sin\sqrt{\frac{k}{l}}\,(t - u)\,du$$

Using $u.dv = uv - \int v\,du$

$u = u \quad ; \quad dv = \sin\sqrt{\frac{k}{l}}\,(t - u)\,du$

$du = 1 \quad ; \quad v = \frac{1}{\sqrt{\frac{k}{l}}}\cos(\sqrt{\frac{k}{l}}\,(t - u))$

$$\Theta = \alpha\,\sqrt{\frac{k}{l}}\,\{[u.\frac{1}{\sqrt{\frac{k}{l}}}\cos\sqrt{\frac{k}{l}}\,(t - u)]_0^t - \int_0^t \frac{1}{\sqrt{\frac{k}{l}}}\quad \cos\sqrt{\frac{k}{l}}\ (t - u)\ du\}$$

Integrating by parts

$$\Theta = \alpha\,\sqrt{\frac{k}{l}}\,\{[t.\frac{1}{\sqrt{\frac{k}{l}}}\cos\sqrt{\frac{k}{l}}\,(t - t)] - \{[0.\frac{1}{\sqrt{\frac{k}{l}}}\cos\sqrt{\frac{k}{l}}\,(t - 0)]\ - \int_0^t \frac{1}{\sqrt{\frac{k}{l}}}\ \cos\sqrt{\frac{k}{l}}\ (t - u)\ du\}$$

$$\Theta = \alpha \sqrt{\frac{k}{l}} \left\{ \left[\frac{t}{\sqrt{\frac{k}{l}}} \right] - \left[\frac{1}{-1} \frac{l}{k} sin \sqrt{\frac{k}{l}} (t-u) \right]_0^t \right\}$$

$$\Theta = \alpha \sqrt{\frac{k}{l}} \left\{ \left[\frac{t}{\sqrt{\frac{k}{l}}} \right] + \left[\frac{l}{k} sin \sqrt{\frac{k}{l}} (t-u) \right]_0^t \right\}$$

$$\Theta = \alpha \sqrt{\frac{k}{l}} \left\{ \sqrt{\frac{l}{k}} t + \left(\left[\frac{l}{k} sin \sqrt{\frac{k}{l}} (t-t) \right] - \left[\frac{l}{k} sin \sqrt{\frac{k}{l}} (t-0) \right] \right) \right\}$$

$$\Theta = \alpha \sqrt{\frac{k}{l}} \left\{ \sqrt{\frac{l}{k}} t - \frac{l}{k} sin \sqrt{\frac{k}{l}}\, t \right\}$$

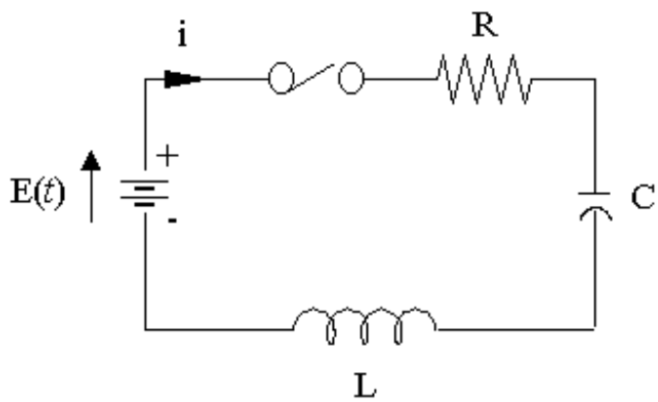
$$\Theta = \alpha \sqrt{\frac{k}{l}} \cdot \sqrt{\frac{l}{k}} t - \alpha \sqrt{\frac{k}{l}} \; \frac{l}{k} sin \sqrt{\frac{k}{l}}\, t$$

$$\Theta(t) = \alpha t - \alpha \sqrt{\frac{l}{k}} sin \sqrt{\frac{k}{l}} t$$

This is the required turn at any time.

4.2 LAPLACE TRANSFORM IN SIMPLE ELECTRIC CIRCUIT

Consider an electric circuit considering of a resistance R, Inductance L, a condenser capacity C and electromotive power of voltage E in a series .A switch is also connected in the circuit.



Then by kirchoff’s law, we get

$$L\frac{dI}{dt} + RI + \frac{Q}{c} = E.$$

Example 1: An inductance of 2 hendry ,a resistor of 16 ohms and capacitor of 0.02 farad are connected in series with an emf of 300 volts, a at **t=0**. To find the charge and current at any time t>0.

Solution:

Let Q and I be instantaneous charge and current respectively at time **t**

Then by kirchoff's law

$$L\,\frac{dI}{dt} + RI + \frac{Q}{C} = E$$

$$2\frac{d^2Q}{dt^2} + 16\,\frac{dQ}{dt} + 50Q = E \qquad \{\text{since } I = \frac{dQ}{dt}\}$$

$$\frac{d^2Q}{dt^2} + 8\frac{dQ}{dt} + 25Q = 150$$

Applying Laplace Transform on both sides,

$$L\left[\frac{d^2Q}{dt^2}\right] + 8L\left[\frac{dQ}{dt}\right] + 25L[Q] = L\,[150]$$

$$\{\,s^2L[Q] - sQ(0) - Q'(0)\} + 8\{sL[Q] - Q(0)\} + 25L[Q] = 150L[1]$$

$$s^2L[Q] + 8\,sL[Q] + 25L[Q] = \frac{150}{s}$$

$$(\,s^2 + 8\,s + 25)L[Q] = \frac{150}{s}$$

$$L[Q] = \left[\frac{150}{s(s^2 + 8\,s + 25)}\right]$$

By method of partial fraction ,

$$I_Q = \frac{150}{s(s^2+8s+25)} = \frac{A}{S} + \frac{BS+C}{s^2+8s+25}$$

$$\frac{150}{s(s^2+8s+25)} = \frac{A(s^2+8s+25)+(BS+C)s}{S(s^2+8s+25)}$$

$$150 = AS^2 + 8AS + 25A + BS^2 + CS$$

$$150 = S^2(A+B) + S(8A+C) + 25A$$

$$A+B = 0 \qquad (4.2.\ 1)$$

$8A+B = 0$ (4.2. 2)

$25A = 150$ (4.2. 3)

Solve equation (4.2. 3)

$$25A = 150$$

$$A= 6$$

$$A = 6 \text{ in } (4.2.\ 1)$$

$$A+B= 0$$

$$6+B = 0$$

$$B = -6$$

$$A = 6 \text{ in } (4.2.\ 2)$$

$$8A+C = 0$$

$$8(6)+C = 0$$

$$48 +C = 0$$

$$C = -48$$

$$Q = L^{-1}\left(\frac{A}{S} + \frac{BS+C}{S^2+8S+25}\right)$$

$$Q = L^{-1}\left[\frac{6}{S} - \frac{6S+48}{(s^2+8s+25)}\right]$$

$$Q = 6L^{-1}\left[\frac{1}{s}\right] - L^{-1}\left[\frac{6(s+4)}{(s+4)^2+9}\right] L^{-1}\left[\frac{24}{(s+4)^2+9}\right]$$

Using Shifting property

$$Q = 6\text{-}6e^{-4t}cos\ 3t - 8e^{-4t}sin\ 3t$$

And $$I = \frac{dQ}{dt} = -6\left[e^{-4t}(3sin3t) + cos3t(-4e^{-4t})\right]$$

$$-8[e^{-4t}(3cos3t) + (sin3t)(-4e^{-4t})]$$

$$I = \frac{dQ}{dt} = 18e^{-4t}\text{sin3t} + 24e^{-4t}\ \text{cost} - 24e^{-4t}\ \text{cost} + 32\ \text{sin}3te^{-4t}$$

$$I = \frac{dQ}{dt} = 50e^{-4t} sin\ 3t$$

This is required expression for charge and current at any time t>0.

Example 2 :

In the theory of electrical circuits, the current flow in a capacitor proportional to the capacitance and the rate of change in the electrical potential in SI units.

Symbolically, this is expressed by the differential equation

$$i = C\frac{dv}{dt}$$

where C is the capacitance (in frads) of the capacitor, i = i(t) is the electric current (in ampheres) through the capacitor as a function of time and v = v(t) is the voltage (in volts) across the terminals of the capacitor, also a function of time.

Taking Laplace transform,

$$I\ (s) = C\ [\ s\ V(s) - V_0]$$

Where I (s) = L [i (t)],

$$V(s) = L[v(t)],\ V_0 = V(t)\ \text{ when, } t = 0$$

$$V(s) = \frac{I\ (s)}{sC} + \frac{V_0}{s} \qquad (4.2.\ 4)$$

The definition of the complex impedance Z (in ohms) is the ratio of the complex voltage V divided by the complex current I while holding the initial state V_0 at zero:

$$Z(s) = \frac{V(s)}{I(s)}\ \text{ when, } V_0 = 0$$

Using this definition and equation (4.2. 4),

$$Z(s) = \frac{1}{sC}$$

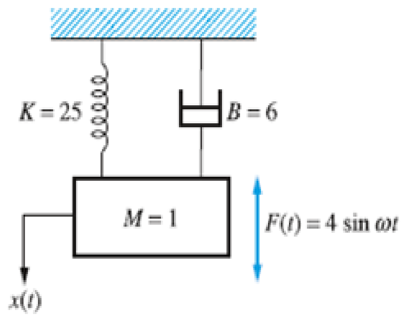
This is the correct expression for the complex impedance of a capacitor.

4.3 APPLICATION IN MECHANICAL ENGINEERING

Vibrating mechanical system:

In examining the suspension system of the car the important elements in the system are the mass of the car and the springs and damper used to connect to the body of the car to the suspension links. Mechanical translational systems may be used to model many situations, and involve three basic elements: **masses** (having mass M, measured in kg), **springs** (having spring stiffness K, measured in Nm^{-1}) and **dampers** (having damping coefficient B, measured in Nsm^{-1}). The associated variables are **displacement** x(t) (measured in m) and force F(t) (measured in N).

Consider Mass-spring-damper system,



From the above diagram we have,

K=25, B=6, M=1, F (t) = 4sinωt

Mass-spring-damper system can be modeled using Newton's and Hooke's law. Therefore the differential equation representing the above system given by

$$M\frac{d^2x}{dt^2} + B\frac{dx}{dt} + K\,x = F\,(t)$$

$$\frac{d^2x}{dt^2} + 6\,\frac{dx}{dt} + 25x = 4\sin\omega t \qquad (4.3.1)$$

Taking Laplace transforms throughout in (4.3.1) gives,

$$L\,[x''\,(t)] + 6L\,[x'\,(t)] + 25L\,[x(t)] = L\,[4\sin\omega t]$$

Using the derivative property of Laplace transform, we get

s^2L [x(t)] – sx'(0) + 6 sL [x(t)] – 6 x(0) +25L [x(t)] – x(0) = L [4sinωt]

s^2L [x(t)] + 6 sL [x(t)] +25L [x(t)] = sx'(0) + 6 x(0) + x(0) + L [4sinωt]

$$(s^2 + 6s + 25)\,L\,[x(t)] = [sx'(0) + x(0) + 6x(0)] + \frac{4\omega}{(s^2+\omega^2)}$$

(since, L [sinωt] = $\frac{\omega}{s^2+\omega^2}$)

And we take the initial conditions $x(0) = 0 = x'(0)$

$$L\,[x(t)] = \frac{4\omega}{(s^2+\omega^2)\,(s^2+6s+25)}$$

Which, on resolving into partial fractions (with $\omega = 2$), leads to

$$L\,[x(t)] = \frac{8}{(s^2+4)(s^2+6s+25)} = \frac{As+B}{(s^2+4)} + \frac{Cs+D}{(s^2+6s+25)}$$

Taking inverse Laplace transforms gives the required response

[L^{-1} [F(s)] = f (t) is the inverse Laplace transforms]

By this condition L [x(t)] becomes,

$$x\,(t) = L^{-1}\left[\frac{8}{(s^2+4)(s^2+6s+25)}\right]$$

$$\left[\frac{8}{(s^2+4)(s^2+6s+25)}\right] = \frac{As+B}{(s^2+4)} + \frac{Cs+D}{(s^2+6s+25)}$$

$8 = As + B(s^2 + 6s + 25) + Cs + D(s^2 + 4)$

$8 = (As^3 + 6As^2 + 25As + Bs^2 + 6Bs + 25B) + (Cs^3 + 4Cs + Ds^2 + 4D)$

$8 = (A+C)\,s^3 + (6A+B+D)\,s^2 + (25A+6B+4C)s + (25B+4D)$

Equating the coefficients of s^3,

$$A+C = 0$$

Equating the coefficients of s^2,

$$6A+B+D = 0$$

Equating the coefficients of s,

$$25A+6B+4C = 0$$

Equating constant terms,

$$25B+4D = 8$$

By solving above equations we get $x(t)$

$x(t)= \frac{4}{195}(7\sin 2t - 4\cos 2t) + \frac{2}{195}\, e^{-3t}(8\cos 4t - \sin 4t).$

CONCLUSION

In this project, an attempt is made to study about the application of Laplace transform in various fields.

- In the population growth model, the population at time t was obtained.
- In the nuclear physics model, expression for amount of radioactive substances was found.
- In the mechanics model, velocity of the particle at time t was found.
- In theory of automatic control, angle of turn of enemy's aircraft at time t was found.
- In electric circuit model, expression for charge and current at any time t were obtained.
- In mass spring damper system, displacement of mass-spring-damper system of the car at time t was found.